Use of MATLAB to Solve Systems of Linear Equations

Department of Electrical Engineering

Contents

- **Part 1 : Basics**
  - Matrices
  - Size of Matrices
  - Matrix Operations: Addition/Subtraction and Scalar Multiplication/Division
  - Determinant of a Matrix
  - Inverse of a Matrix
  - Matrices Multiplication and Division
  - Eigenvalues and Eigenvectors
- **Part 2 : System of Linear Equations**
  - Bungee Jumpers Problem
  - Currents and Voltages in Circuits

Part 1 : Basics

Basics

- **Matrices**

"A Matrix is a set of numbers."

<u>Examples</u>

$$\begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 12 & 4 & -100 \end{bmatrix}, \begin{bmatrix} 5 & 2 & 3 \\ 6 & 1 & 3 \\ 4 & 11 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix}$$

Basics Cont.

❖ **Size of Matrices**

The size of a Matrix is defined by the number of rows and the number of columns.

<u>Exercises</u> (use the Matlab function "size")

a) $\text{size}\left(\begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}\right) = ??$

b) $\text{size}\left(\begin{bmatrix} 12 & 4 & -100 \end{bmatrix}\right) = ??$

c) $\text{size}\left(\begin{bmatrix} 5 & 2 & 3 \\ 6 & 1 & 3 \\ 4 & 11 & 9 \end{bmatrix}\right) = ??$

d) $\text{size}\left(\begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix}\right) = ??$

Basics Cont.

❖ **Matrices Operations: Addition/Subtraction $A \pm B$**

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \pm \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{bmatrix}$$

Exercises

a) $\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix} = ??$

b) $[5 \quad 3 \quad -2] - [0 \quad -1 \quad 99] = ??$

Basics Cont.

❖ **Matrices Operations: Scalar Multiplication/Division $A \times c,\ A/c$**

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times c = \begin{bmatrix} c\ a_1 \\ c\ a_2 \\ c\ a_3 \end{bmatrix}$$

$$\begin{bmatrix} a & b & d \\ e & f & g \end{bmatrix} / c = \begin{bmatrix} a/c & b/c & d/c \\ e/c & f/c & g/c \end{bmatrix}$$

Exercises

a) $4 \times \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = ??$

b) $\begin{bmatrix} 5 & 3 & -2 \end{bmatrix} \times (-2) = ??$

c) $\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} /3 \quad = ??$

d) $\begin{bmatrix} 1 & 0 & 3 \\ 4 & 5 & 6 \end{bmatrix} /(-1) \quad = ??$

Basics Cont.

❖ **Determinant of a Matrix**

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= ad - bc$$

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Basics Cont.

❖ **Determinant of a Matrix**

<u>Exercises</u> (use the Matlab function "det")

a) $\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = ??$

b) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = ??$

c) $\begin{vmatrix} 1 & 2 & -3 & 1 \\ 2 & 3 & 4 & 2 \\ 5 & 6 & 11 & 3 \\ 6 & -1 & 12 & 4 \end{vmatrix} = ??$

Basics Cont.

- **Inverse of a Matrix**

For a Matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse can be found as:

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

<u>Exercises</u> (use the Matlab function "inv")

a) $Inv\left(\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\right)$ **= ??**

b) $\begin{bmatrix} 1 & 2 & 5 \\ -2 & 5 & 7 \\ 3 & 8 & 9 \end{bmatrix}^{-1}$ **= ??**

Basics Cont.

❖ **Matrices Multiplication and Division**

Multiplication (dot product) between two matrices has the condition that:

columns of the 1st matrix must equal the rows of the 2nd matrix, and the result will have the same number of rows as the 1st matrix, and the same number of columns as the 2nd matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$$

Basics Cont.

❖ **Matrices Multiplication and Division**

Exercises

a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = ??$

b) $\begin{bmatrix} 3 & 4 & 2 \end{bmatrix} \times \begin{bmatrix} 13 & 9 & 7 & 15 \\ 8 & 7 & 4 & 6 \\ 6 & 4 & 0 & 3 \end{bmatrix} = ??$

Basics Cont.

- **Eigenvalues and Eigenvectors**

The basic equation is $Av = \lambda v$

The eigenvalue λ tells whether the eigenvector v is stretched or shrunk or reversed or left unchanged when it is multiplied by A. λ can be found to be 2, 0.5, 1, or -1. The eigenvalue could be zero, then $\boldsymbol{Av = 0}$ means that this eigenvector v is in the null-space.

λ is found by solving the equation det(A- λ I) = 0, then v is found by solving the basic equation.

Basics Cont.

❖ **Eigenvalues and Eigenvectors**

<u>Exercises</u> (use the Matlab function "[V, L] = eig(A)")

a) $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ = ??

b) $\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ = ??

Part 2 : System of Linear Equations

Systems of Linear Equations

<u>**This title consists of two parts:**</u>

a) **Linear Equations**

b) **System of Equations**

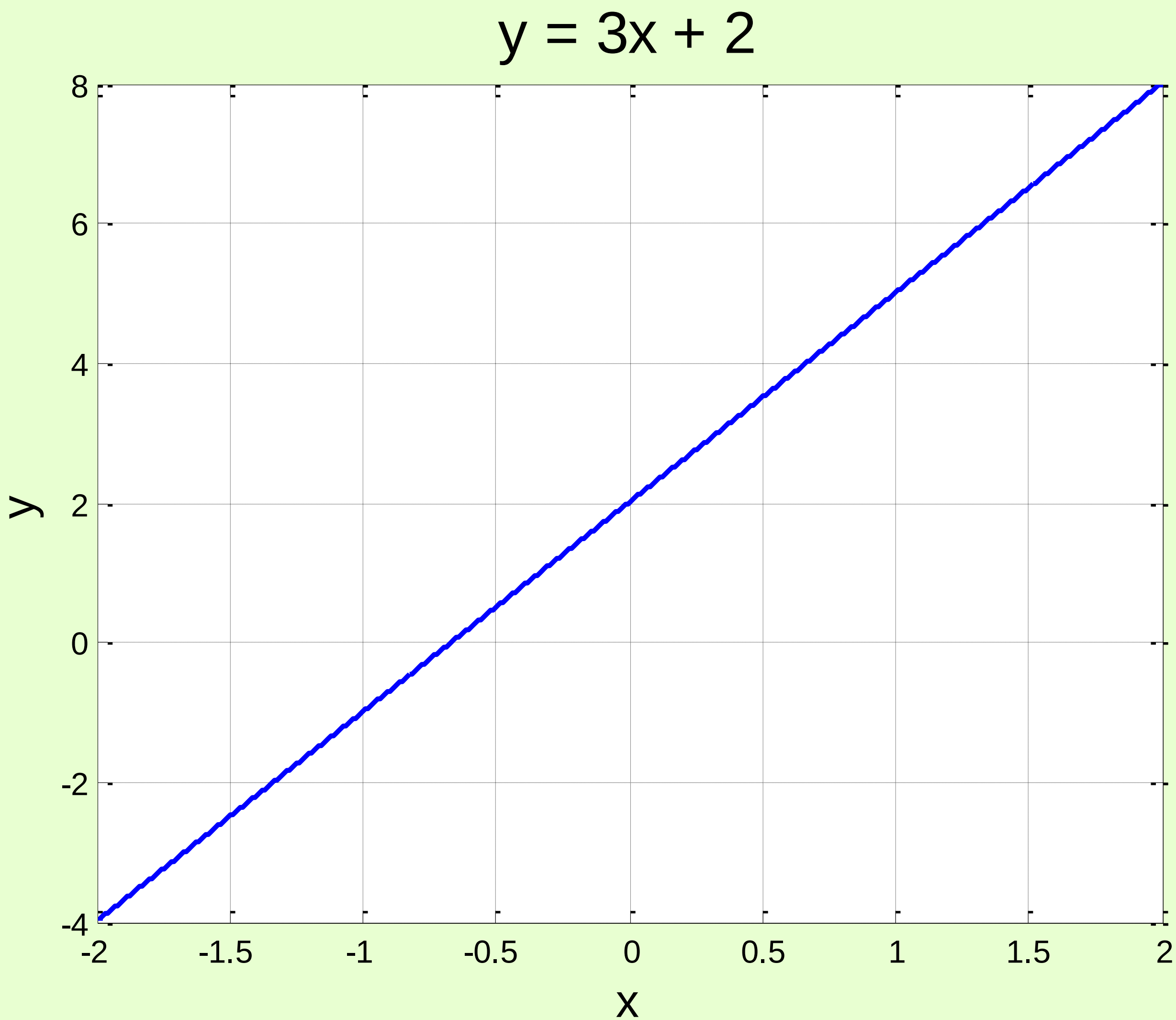
Systems of Linear Equations Cont.

Linear equations are represented by

$$y = ax + b$$

For example: $y = 3x + 2$

Systems of Linear Equations Cont.



Systems of Linear Equations Cont.

System of three linear equations is represented by:

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_1 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_2 \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + b_3 \end{aligned}$$

The matrix notation is:

$$Y = AX + B$$

For example:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Systems of Linear Equations Cont.

Solving a system of linear equations can be basically done by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} \left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right)$$

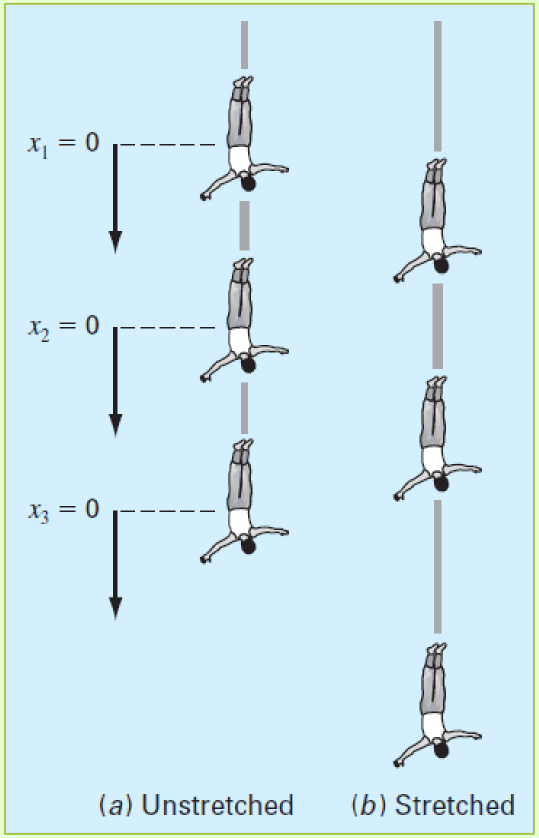
Bungee Jumpers Problem

Problem Statement:

Suppose that three jumpers are connected by bungee cords. Figure *a* shows them being held in place vertically so that each cord is fully extended but un-stretched. We can define three distances, $x1$, $x2$, and $x3$, as measured downward from each of their un-stretched positions.

After they are released, gravity takes hold and the jumpers will eventually come to the equilibrium positions shown in Figure b.

Suppose that you are asked to compute the displacement of each of the jumpers, assuming that each cord behaves as a linear spring and follows Hooke's law.



Bungee Jumpers Problem Cont.

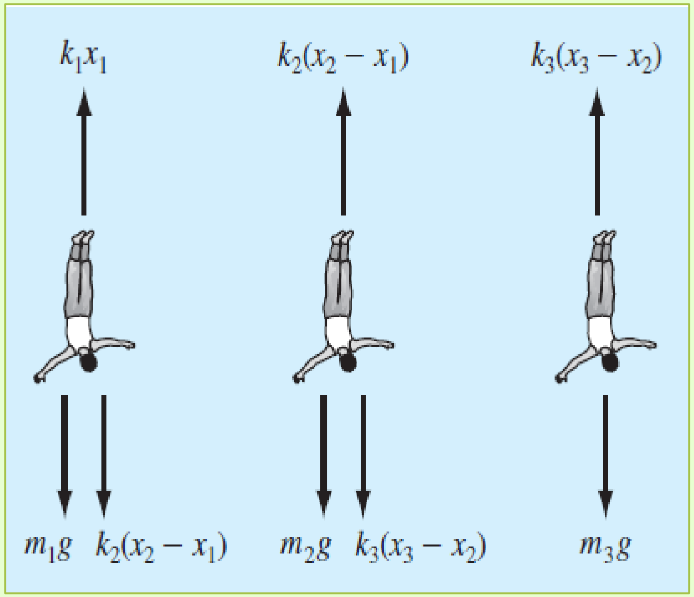
Using Newton's second law, a steady-state force balances can be written for each jumper as:

$$\begin{aligned}
(k_1 + k_2)x_1 \quad -k_2x_2 \quad\quad\quad &= m_1g \\
-k_2x_1 \quad +(k_2 + k_3)x_2 \quad -k_3x_3 &= m_2g \\
-k_3x_2 \quad +k_3x_3 &= m_3g
\end{aligned}$$

Where m_i = the mass of jumper i (kg)

k_j= the spring constant for cord j (N/m)

x_i= the displacement of jumper i



Bungee Jumpers Problem Cont.

Jumper	Mass (Kg)	Spring Constant (N/m)	Un-stretched Cord Length (m)
Top (1)	60	50	20
Middle (2)	70	100	20
Bottom (3)	80	50	20

$$\begin{bmatrix} m_1 g \\ m_2 g \\ m_3 g \end{bmatrix} = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 588.6 \\ 686.7 \\ 784.8 \end{bmatrix} = \begin{bmatrix} 150 & -100 & 0 \\ -100 & 150 & -50 \\ 0 & -50 & 50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Bungee Jumpers Problem Cont.

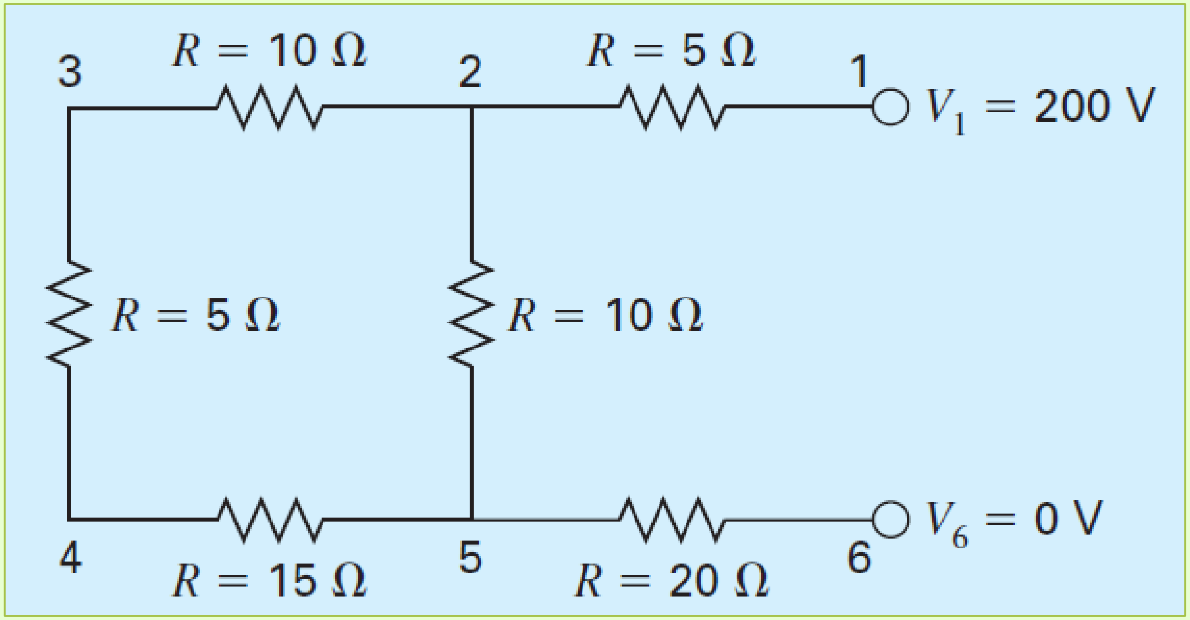
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 150 & -100 & 0 \\ -100 & 150 & -50 \\ 0 & -50 & 50 \end{bmatrix}^{-1} \begin{bmatrix} 588.6 \\ 686.7 \\ 784.8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 41.2020 \\ 55.9170 \\ 71.6130 \end{bmatrix}$$

Currents and Voltages in Circuits

Problem Statement:

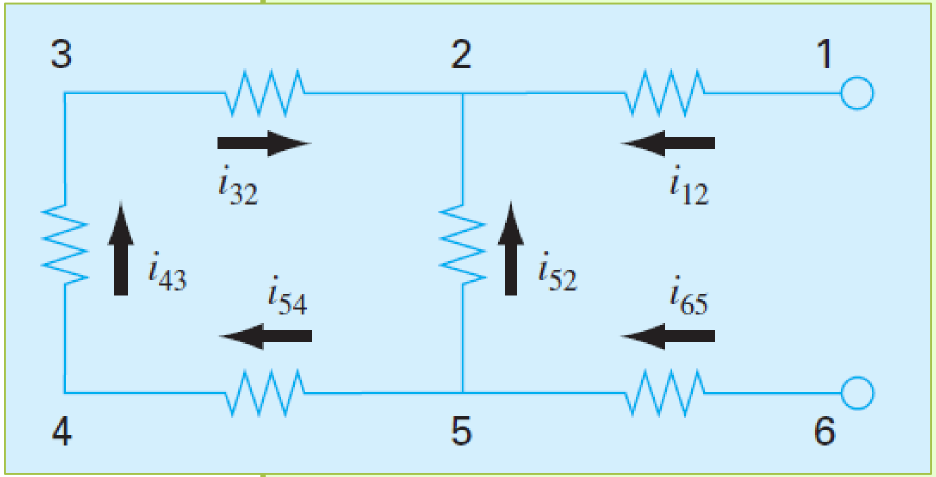
Find the values of all currents running through this circuit



Currents and Voltages in Circuits

By applying the voltage and current rules:

$$i_{12} + i_{52} + i_{32} = 0$$
$$i_{65} - i_{52} - i_{54} = 0$$
$$i_{43} - i_{32} = 0$$
$$i_{54} - i_{43} = 0$$
$$-15i_{54} - 5i_{43} - 10i_{32} + 10i_{52} = 0$$
$$-20i_{65} - 10i_{52} + 5i_{12} = 200$$



$$\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{200} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{10} & \mathbf{-10} & \mathbf{0} & \mathbf{-15} & \mathbf{-5} \\ \mathbf{5} & \mathbf{-10} & \mathbf{0} & \mathbf{-20} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{i_{12}} \\ \boldsymbol{i_{52}} \\ \boldsymbol{i_{32}} \\ \boldsymbol{i_{65}} \\ \boldsymbol{i_{54}} \\ \boldsymbol{i_{43}} \end{bmatrix}$$

Currents and Voltages in Circuits

$$\begin{bmatrix} i_{12} \\ i_{52} \\ i_{32} \\ i_{65} \\ i_{54} \\ i_{43} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 10 & -10 & 0 & -15 & -5 \\ 5 & -10 & 0 & -20 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 200 \end{bmatrix}$$

$$\begin{bmatrix} i_{12} \\ i_{52} \\ i_{32} \\ i_{65} \\ i_{54} \\ i_{43} \end{bmatrix} = \begin{bmatrix} 6.1538 \\ -4.6154 \\ -1.5385 \\ -6.1538 \\ -1.5385 \\ -1.5385 \end{bmatrix}$$

Thank you for your attention …

Any Questions ?