Math help desk- session (3)

Department of Electrical Engineering

OVERVIEW

- Determinants
- Inverse of a matrix
- Solving systems of linear equations.
- Eigenvalues and eigenvectors

Determinant of a Matrix

- Determinant is associated with **Square matrix**

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ → $\text{Det}(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

- The determinant of a matrix may be negative , positive or zero.
- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; det(A)= ad−bc.

- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$; det(A)= (aei+bfg+cdh)-(ceg+bdi+afh)

EXAMPLES

- Ex:1 A= $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$;;; |A|=?
- Ans= -2

- Ex:2 A= $\begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}$;;; |A|=?

- Ans: A) -13 B) -7
  C) 7 D) 13

- Ex:3 A= $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 2 & -1 \\ -5 & 3 & -2 \end{bmatrix}$;;; det(A)=?

Solution:

$|A| =$

$2[2 \times (-2) - (-1) \times 3] - (-3)[4 \times (-2) - (-1) \times (-5)] + 1[4 \times 3 - 2 \times (-5)]$

$= 2[(-4) - (-3)] + 3[(-8) - 5] + 1[12 - (-10)]$

$= 2 \times (-1) + 3 \times (-13) + 1 \times 22$

$= -2 - 39 + 22$

$= -19$

Ex:4 $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \ldots$

- $|A| = 0$ ⇒ A is SINGULAR MATRIX

- ***ie; If the determinant is zero, the matrix is singular…..***

Adjoint of a Matrix

- The matrix formed by taking the transpose of the cofactor matrix of the original matrix.
- The adjoint of matrix A is often written **adj A.**
- Ex:1 Find adj(A)? A= $\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix}$
- Step 1: Find the minor of each element.
- Minor of a11,, M11= $\begin{matrix} a22 & a23 \\ a32 & a33 \end{matrix}$ =(a22* a33) – (a23*a32)
- M12= $\begin{matrix} a21 & a23 \\ a31 & a33 \end{matrix}$ =(a21*a33)-(a23*a31)

- Matrix of Minor

$$\begin{bmatrix} M11 & M12 & M13 \\ M21 & M22 & M23 \\ M31 & M32 & M33 \end{bmatrix}$$

- Step 2: Form Co-factor Matrix

$$\begin{bmatrix} +M11 & -M12 & +M13 \\ -M21 & +M22 & -M23 \\ +M31 & -M32 & +M33 \end{bmatrix}$$

- Complete cofactor matrix and then find the transpose of the matrix.

- Step 3: Adj(A)= (Co-factor Matrix)^T

Examples….

- Find Cofactor matrix of $A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$

Inverse of a Matrix

- **What is the Inverse of a Matrix?**

- $A^{-1} = \text{Adj}(A) \,/\, |A|$

- When you **multiply a Matrix** by its **Inverse** you get the **Identity Matrix** (I)

- $A \times A^{-1} = \mathbf{I}$

- $$\mathbf{I} = \begin{matrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{matrix}$$

- The Inverse of A is A^{-1} only when:
  - $\mathbf{A \times A^{-1} = A^{-1} \times A = I}$
- ***Sometimes there is no Inverse…….??????***
- *If the determinant is zero, the matrix is singular and does not have an inverse.*

How do we calculate the Inverse?

- $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- EX:1 $A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$,,, Find A^{-1}

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

Examples

- **What is the Inverse of a Matrix?**
- $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$
- Ans: $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$
- **What is the Inverse of a Matrix?**
- $A = \begin{bmatrix} 7 & 4 \\ 2 & 1 \end{bmatrix}$
- Ans: $\begin{bmatrix} -1 & 4 \\ 2 & -7 \end{bmatrix}$

Inverse of a 3X3 Matrix

- Step 1: calculating the Matrix of Minors,
- Step 2: Then turn that into the Matrix of Cofactors,
- Step 3: Form the Adjoint (Adjugate) matrix
- Step 4: Multiply that by 1/Determinant

Inverse of a 3X3 Matrix

- Ex:1 $$A=\begin{bmatrix}1 & 2 & 3\\0 & 4 & 5\\1 & 0 & 6\end{bmatrix}$$

Inverse of a 3X3 Matrix

- $A = \begin{bmatrix} 5 & 3 & 7 \\ 2 & 4 & 9 \\ 3 & 6 & 4 \end{bmatrix}$

- Ans:

$$\begin{pmatrix} \frac{2}{7} & -\frac{30}{133} & \frac{1}{133} \\ -\frac{1}{7} & \frac{1}{133} & \frac{31}{133} \\ 0 & \frac{3}{19} & -\frac{2}{19} \end{pmatrix}$$

0.2857	-0.2256	0.0075
-0.1429	0.0075	0 .2331
0	0.1579	-0.1053

Solution of simultaneous equations using the inverse matrix (MATRIX ALGEBRA)

Linear Equation

- A system of equations in which each equation is linear.

- For any linear system,
  - There is only one solution, Or
  - there are infinitely many solutions (consistent), Or
  - there are no solutions (inconsistent).

- It is possible to represent a system of simultaneous linear equations as a matrix equation.

Linear Equation

- We have one linear equation $Ax = B$; x is unknown and A & B are constants,,, then there are just three possibilities.
- **Conditions**

1. $A \neq 0$ then $x=B/A = A^{-1} B$ .Then the equation $ax = b$ has a ***unique solution*** for x.

2. $A = 0$, $B = 0$ then the equation $Ax = B$ becomes $0 = 0$ and any value of x will do. There are ***infinitely many solutions*** to the equation $Ax = B$.

3. $A = 0$ and $B \neq 0$ then $Ax = B$ becomes $0 = b$ which is a contradiction. In this case the equation $Ax = B$ has *no solution* for x.

Linear Equation

- Ex:1 $2x_1 + 3x_2 = 5$
  $x_1 - 2x_2 = -1.$

Solution: We have to form the equations as below

$$AX = B.$$

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \; ; X = \begin{bmatrix} x1 \\ x2 \end{bmatrix} \; ; B = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

then the solution is; $X = A^{-1} * B$

Step 1: check whether A^{-1} *exists or not……(* $|A| \neq 0$*)*

Step 2: Find A^{-1} *and solve the eqn*

Ans: $x1 = 1, x2 = 1.$

1.Solve the following using the inverse matrix approach:

(a) $3x - 2y = 17$

$5x + 3y = 3$

Step :1 $A = \begin{bmatrix} 3 & -2 \\ 5 & 3 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \end{bmatrix}$; $B = \begin{bmatrix} 17 \\ 3 \end{bmatrix}$

Step 2: $|A| = 9-(-10) = 19$; adj(A)= $\begin{bmatrix} 3 & 5 \\ -2 & 3 \end{bmatrix}$

inv(A)= $\begin{bmatrix} 3/19 & 5/19 \\ -2/19 & 3/19 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix}$ = inv(A)*B $= \begin{bmatrix} 3/19 & 5/19 \\ -2/19 & 3/19 \end{bmatrix} * \begin{bmatrix} 17 \\ 3 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

1. Solve the following using the inverse matrix approach:

(a) $2x - 3y = 1$

$4x + 4y = 2$

Step :1 $A = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Step 2: $|A| = 20$; $A^{-1} = 1/20 \left\{ \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} \right\}$;

$$X = \left\{ \begin{bmatrix} 4/20 & 3/20 \\ -4/20 & 2/20 \end{bmatrix} \right\} * \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

(b) $2x - 5y = 2$

$-4x + 10y = 1$

(A^{-1} does not exist.)

(c) $6x - y = 0$

$2x - 4y = 1$

2. Solve the following equations using matrix methods:

(a) $2x1 + x2 - x3 = 0$

$x1 + x3 = 4$

$x1 + x2 + x3 = 0$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \; ; X = \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} \; ; B = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

$x1 = 8/3, x2 = -4, \quad x3 = 4/3$

(b) $x1 - x2 + x3 = 1$

$-x1 + x3 = 1$

$x1 + x2 - x3 = 0$

$x1 = ½ \; ; \; x2 = ½ \; ; \; x3 = 1$

Eigen values & Eigen vectors

- Eigenvalues are a special set of scalars associated with a linear system of equations and known as characteristic roots.

- The basic equation is $Av = \lambda v$; λ is an eigenvalue of A and v is the eigen vector of A

- If $\lambda = 0$,,,Av=0v and then eigen vector "v" is called "null space".

λ is called Eigen value of a matrix "A" iff,, $\det(A - \lambda I) = 0$

- **Example 1**: Determine the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

- First, form the matrix $A - \lambda I$: $\begin{bmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{bmatrix}$

- Take the det($A - \lambda I$): = $(1-\lambda)(-4-\lambda)$ - $[(-2)(3)]=0$

  = $\lambda^2 + 3\lambda + 2$; Which is called "CHARACTERISTIC POLYNOMIAL"

- The solutions of the characteristic equation, det($A - \lambda I$) = 0, are the eigenvalues of A: $\lambda^2 + 3\lambda + 2 = 0$

- $(\lambda+1) * (\lambda+2)=0$ → λ = -1 or -2

EXAMPLE 1: Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

SOLUTION:

- In such problems, we first find the **eigenvalues** of the matrix.

FINDING EIGENVALUES

- To do this, we find the values of λ which satisfy the **characteristic equation** of the matrix A, namely those values of λ for which

$$\det(A - \lambda I) = 0,$$

where I is the 3×3 **identity matrix**.

- Form the matrix $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{pmatrix}.$$

Notice that this matrix is just equal to A with λ subtracted from each entry on the main diagonal.

- Calculate $\det(A - \lambda I)$:

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)\begin{vmatrix} -5-\lambda & 3 \\ -6 & 4-\lambda \end{vmatrix} - (-3)\begin{vmatrix} 3 & 3 \\ 6 & 4-\lambda \end{vmatrix} + 3\begin{vmatrix} 3 & -5-\lambda \\ 6 & -6 \end{vmatrix} \\ &= (1-\lambda)\left((-5-\lambda)(4-\lambda) - (3)(-6)\right) + 3(3(4-\lambda) - 3\times 6) + 3(3\times(-6) - (\\ &= (1-\lambda)(-20 + 5\lambda - 4\lambda + \lambda^2 + 18) + 3(12 - 3\lambda - 18) + 3(-18 + 30 + 6\lambda) \\ &= (1-\lambda)(-2 + \lambda + \lambda^2) + 3(-6 - 3\lambda) + 3(12 + 6\lambda) \\ &= -2 + \lambda + \lambda^2 + 2\lambda - \lambda^2 - \lambda^3 - 18 - 9\lambda + 36 + 18\lambda \\ &= 16 + 12\lambda - \lambda^3. \end{aligned}$$

- Therefore

$$\det(A - \lambda I) = -\lambda^3 + 12\lambda + 16.$$

* Taking $\lambda = 4$, we find that $4^3 - 12.4 - 16 = 0$.
* Now factor out $\lambda - 4$:

$$(\lambda - 4)(\lambda^2 + 4\lambda + 4) = \lambda^3 - 12\lambda^2 + 16.$$

* Solving $\lambda^2 + 4\lambda + 4$ by formula[1] gives

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4.1.4}}{2} = \frac{-4 \pm 0}{2},$$

  and so $\lambda = -2$ (a repeated root).

- Therefore, the eigenvalues of A are $\lambda = 4, -2$. ($\lambda = -2$ is a repeated root of the characteristic equation.)

6.2 Eigenvalues: examples

Example 1: Find the eigenvalues of

$$A=\begin{bmatrix}2 & -12\\ 1 & -5\end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix}\lambda-2 & 12\\ -1 & \lambda+5\end{vmatrix} = (\lambda-2)(\lambda+5)+12$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$$

two eigenvalues: $-1, -2$

Example 2: Find the eigenvalues of

$$A=\begin{bmatrix}2 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix}\lambda-2 & -1 & 0\\ 0 & \lambda-2 & 0\\ 0 & 0 & \lambda-2\end{vmatrix} = (\lambda-2)^3 = 0$$

$\lambda = 2$ is an eigenvector of multiplicity 3.

Example 2: Find the eigenvectors of the 2 by 2 matrix

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$|\mathbf{A} - \lambda \cdot \mathbf{I}| = \left| \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} \right| = \lambda^2 + 3\lambda + 2 = 0$$

λ_1=-1, λ_2=-2

Find eigen vectors of those eigen values.

$$\mathbf{A} \cdot \mathbf{v}_1 = \lambda_1 \cdot \mathbf{v}_1$$

$$(\mathbf{A} - \lambda_1) \cdot \mathbf{v}_1 = 0$$

$$\begin{bmatrix} -\lambda_1 & 1 \\ -2 & -3-\lambda_1 \end{bmatrix} \cdot \mathbf{v}_1 = 0$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \cdot \mathbf{v}_1 = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \cdot \begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} = 0$$

$$\mathbf{v}_1 = k_1 \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

$$\mathbf{A} \cdot \mathbf{v}_2 = \lambda_2 \cdot \mathbf{v}_2$$

$$(\mathbf{A} - \lambda_2) \cdot \mathbf{v}_2 = \begin{bmatrix} -\lambda_2 & 1 \\ -2 & -3 - \lambda_2 \end{bmatrix} \cdot \mathbf{v}_2 = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_{2,1} \\ v_{2,2} \end{bmatrix} = 0 \quad \text{so}$$

$$2 \cdot v_{2,1} + 1 \cdot v_{2,2} = 0 \quad (\text{or from bottom line: } -2 \cdot v_{2,1} - 1 \cdot v_{2,2} = 0)$$

$$2 \cdot v_{2,1} = -v_{2,2}$$

$$\mathbf{v}_2 = k_2 \begin{bmatrix} +1 \\ -2 \end{bmatrix}$$

- Again, the choice of +1 and -2 for the eigenvector was arbitrary; only their ratio is important.

Example 3.1 (diagonal matrix): find the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

We have

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix}.$$

Hence, we can write the characteristic equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0.$$

This gives $(1-\lambda)(2-\lambda) = 0$, and we find two eigenvalues of the matrix $\mathbf{A}$: $\lambda = 1$ and $\lambda = 2$.

Example 3.2 (triangular matrix): find the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

We have

$$A - \lambda I = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix}.$$

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Matlab Examples........

MatLab : eig(A)

```
>> A=[0 1;-2 -3]
A =
   0   1
  -2  -3

>> [v,d]=eig(A)
v =
   0.7071  -0.4472
  -0.7071   0.8944

d =
  -1   0
   0  -2
```

The eigenvalues are the diagonal of the "d" matrix

The eigenvectors are the columns of the "v" matrix.

Matlab Examples……..

Polynomial method to solve for eigen values

```
>> A=[10  -5; -5  10]
A =
   10   -5
  -5     10

>> p=poly(A)
```

p= 1 -20 75-------- characteristic polynomial.

Roots of "p"

```
  >> d=roots(p)
       d=15;5
```

The eigenvalues are the values of "d" matrix

Matlab Examples........

"eye" function can be used to generate Identity matrix.

>> I=eye(3)

> A=[2 3 -1; -1 2 3; 0 1 2]

A =

2 3 -1

-1 2 3

0 1 2

>> b=[-1 9 5]' or b=[-1; 9; 5];

b =

-1

9

5

>> x=inv(A)*b

x =

-1

1

2

Thank you,,,,,