MATH HELP DESK

Complex Analysis

Department of Electrical Engineering

OVERVIEW

- Introduction
- Power of a complex number
- Root of a complex number
- Elementary Functions
  - Exponential
  - Logarithms
  - Trigonometric and hyperbolic
- MATLAB applications

Complex Number

- General Form → $Z = x + iy$
- x – Re(z) && y – Im(z)
- Complex conjugate - $Z^* = x - iy$
- Polar Form – $z = r(\cos(\theta) + i \sin(\theta))$
- $x = r \cos(\theta)$ && $y = r \sin(\theta)$
- $r^2 = x^2 + y^2$; $\theta = \tan^{-1}(y/x)$.

MATLAB functions

- **real()** gives a number's real part
- **>> z= 2+3i**

  z = 2.0000 + 3.0000i

- **>> real(z) ; ans = 2**
- **imag()** gives a number's imaginary part
- **abs()** gives a number's magnitude
- **>> abs(z); ans =3.6056**
- **angle()** gives a number's angle
- **conj()** gives a number's complex conjugate

MATLAB functions

- A c o m p l e x number o f m a g n i t u d e 11 and p h a s e a n g l e 0 . 7 r a d i a n s

- >> z=11*(cos(0.7)+i*sin(0.7))
- z = 8.4133 + 7.0864i

- >> [abs(z) angle(z)]
- ans = 11.0000 0.7000
- **compass()** to plot a complex number directly:
- z = 3 + 4*i
- >> c o m p a s s (z)

Euler's Formula: Phasor Form

- Euler's Formula states that we can express the trigonometric form as: $e^{i\,\Phi} = \cos\Phi + i\sin\Phi$.

- This is also known as phasor form or Phasors

General Phasor Form: $re^{i\phi}$

More generally we use $re^{i\phi}$ where:

$$re^{i\phi} = r(\cos\phi + i\sin\phi)$$

MATLAB Complex No. Phasor Declaration

```
>> exp( i*(pi/4) )

ans =  0.7071 + 0.7071i

>> [abs(z), angle(z)]

ans =  1.0000     0.7854
```

Power of a complex number

- **De Moivre's Theorem :** If $z = r(\cos(\theta) + i\sin(\theta)$ is a complex number and n is a positive integer, then, $z^n = rn\ (\cos(n\theta) + i\sin(n\theta)$.

Ex:1 compute 6^{th} power of a complex number z= (2+i2)

Step 1: Convert 'z' to polar form : z = r(cos(θ)+ i sin(θ)

$$r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{2}{2}\right) = 45°$$

$$z = 2\sqrt{2}\,(\cos(45) + i\sin(45)$$

Step 2: Find power of z

$$z^6 = (2\sqrt{2})^6(\cos(6*45) + i\sin(6*45) =$$ **-0. 512i**

Power of a complex number

Ex 2: Find: $(-2+3j)^5$

$(-2+3j)^5 = (3.60555 \angle 123.69007°)^5$ (converting to polar form)

$= (3.60555)^5 \angle (123.69007° \times 5)$ (applying deMoivre's Theorem)

$= 609.33709 \angle 618.45035°$

$= -121.99966 - 596.99897j$ (converting back to rectangular form)

$= -122.0 - 597.0j$

Power of a complex number

- **Ex 3 :** $z=(-1+\sqrt{3i})$ **. Find** z^{12} **?.**

- **Ex 4:** If $z = (1 + i)$ find $[(z^4 + 2z^5)/ z']^2$?

$[(z^4 + 2z^5)/ z']^2 \quad = \quad [z^4(1 + 2z)/z')]2$ ----------------(1)

$Z= \sqrt{2}e^{ipi/4}$

$Z'= \sqrt{2}e^{-ipi/4}$

$Z^4= 4\exp(ipi) = -4$; sub in (1)

Ans: 104 exp(i2.33)

Power of a complex number

- $z^6 = (2\sqrt{2})^6 (\cos(6*45) + i\sin(6*45)) =$ **-0. 512i**
- **Ex 2 :** $z=(-1+\sqrt{3i})$ **. Find** z^{12} ?

Use DeMoivre's Theorem to find $(-1 + \sqrt{3}i)^{12}$.

Solution First convert to polar form.

$$-1 + \sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Then, by DeMoivre's Theorem, you have

$$\begin{aligned}(-1 + \sqrt{3}i)^{12} &= \left[2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^{12} \\ &= 2^{12}\left[\cos(12)\frac{2\pi}{3} + i\sin(12)\frac{2\pi}{3}\right] \\ &= 4096(\cos 8\pi + i\sin 8\pi) \\ &= 4096.\end{aligned}$$

Nth Root of a complex number

- Complex Roots : If $z^n = x + yj$ then we expect 'n' complex roots for "z".
- Spacing of n-th roots: the roots will be (360/n) degree apart. That is,
- 2 roots will be 180° apart
- 3 roots will be 120° apart
- 4 roots will be 90° apart
- 5 roots will be 72° apart etc.

Nth Root of a complex number

THEOREM A.5 *n*th Roots of a Complex Number

For a positive integer n, the complex number $z = r(\cos\theta + i\sin\theta)$ has exactly distinct nth roots given by

$$\sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right)$$

where $k = 0, 1, 2, \ldots, n - 1$.

Nth Root of a complex number

Ex: 1 Find the two square roots of $-5+12j$

Step 1: express it in polar form

$$r=\sqrt{(-5)^2+12^2}=13$$

$$\theta=\tan^{-1}(x/y)=\tan-1(5/12)\approx 67.38$$

$$\theta = 180° - 67.38 = 112.62°$$ (since z in 2nd quadrant)

Step 2: Using DeMoivre's Theorem: $(r\angle\theta)^n=(r^n\angle n\theta)$,

$$(-5+12j)^{1/2} = 13^{1/2}\angle(1/2\times 112.62°)$$

$=3.61\angle 56.31°$

$= 2 + 3j$.**---------1st root**

Nth Root of a complex number

- here, n=2, so our roots are 180° apart.

- Add 180° to first root

$$x = 3.61\cos(56.31^\circ + 180^\circ) = 3.61\cos(236.31^\circ) = -2$$

$$y = 3.61\sin(56.31^\circ + 180^\circ) = 3.61\sin(236.31^\circ) = -3$$

- So **second root** is −2−3j.

- **two square roots of −5−12j are 2+3j and −2−3j.**

Nth Root of a complex number

- (i) Find the 4 fourth roots of $81(\cos 60^\circ + j \sin 60^\circ)$

- (ii) Then sketch all fourth roots of $81(\cos 60^\circ + j \sin 60^\circ)$ showing relevant values of r and θ

- sqrt(3-5*i)
- ans = 2.1013 - 1.1897i

Nth Root of a complex number

- **Part (i)**
- There are 4 roots, so they will be θ=90∘ apart.
- I **First root:**
- $81^{1/4}$[cos 460+j sin 460] = 3(cos 15+j sin 15) =**2.90+0.78j**
- II **Second root:**
- Add 90∘ to the first root:
- $3(\cos(15^\circ + 90^\circ) + j\,\sin(15^\circ + 90^\circ)) = 3(\cos 105^\circ + j\,\sin 105^\circ)$
- **= -0.78 + 2.90*j***
- So the first 2 fourth roots of $81(\cos 60^\circ + j\,\sin 60^\circ)$ are:
- 2.90+0.78j and −0.78+2.90j

Elementary Functions of a complex variable

Exponential Form of a Complex Number

- **$re^{j\theta}$** -***r*** is the **absolute value** of the complex number, the same as we had before in the Polar Form; and **θ** is in **radians**.

- Ex1: Express 5(cos135°+j sin 135°) in exponential form.
- r=5 from the ques: express θ=135° in radians.
- 135 degree=135*(π/180) =43π ≈2.36 radians
- So ; 5(cos 135o+j sin135o) $=5e^{43\pi j}$ $\approx 5e^{2.36j}$
- **Matlab code:**
- exp(3+4*i) ; ans = -13.1288 -15.2008

Logarithmic function

Ex 1: Find the natural log of a complex number.

- Z= **re** $^{j\theta}$ **---** ln(z) = u + iv = ln(**re** $^{j\theta}$**)** **=** ln(r) +ln(**e** $^{j\theta}$**)** **=ln(r) + j θ**
-

Matlab code:

- » log(2+3*i) ; ans = 1.2825 + 0.9828i
- » log([1+i, 2+3*i, 1-i]) ; ans = 0.3466 + 0.7854i 1.2825 + 0.9828i 0.3466 - 0.7854i

- **EX:2**
- >> log10(5-3*i); ans = 0.7657 - 0.2347i
- >> log(5-3*i); ans = 1.7632 - 0.5404i

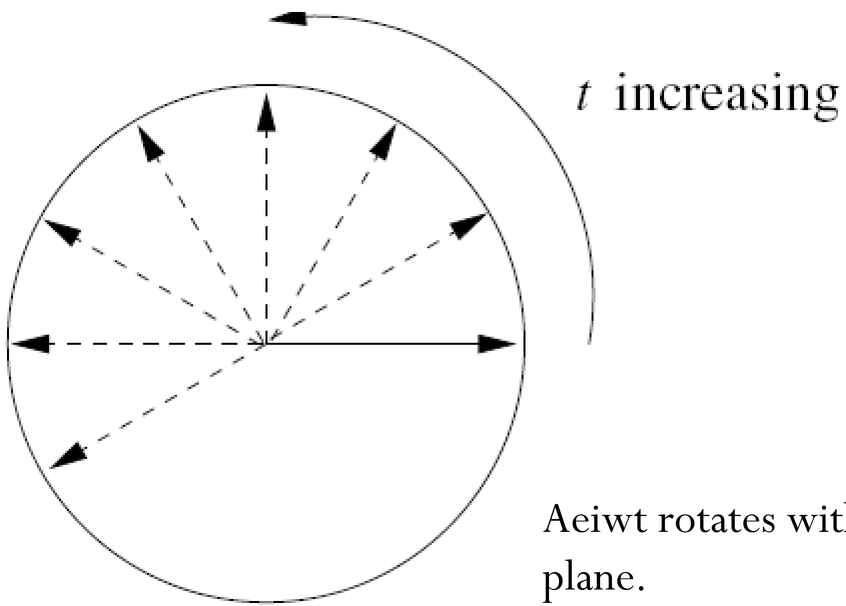
Trig. And hyperbolic function

- » sin(3-5*i)
- ans = 10.4725 +73.4606i
- » cos(-2-8*i)
- ans = -6.2026e+002 -1.3553e+003i
- » tan(0.5-0.3*i)
- ans = 0.4876 - 0.3689i
- » cot(-0.3+0.86*i)
- ans = -0.2746 - 1.3143i

Time varying complex numbers and Phasors

Oscillatory complex variable

- Exponential representation of any complex number: $Z = A\,e^{i\theta}$.
- Consider a complex number z1 which is a function of time (t)
- $\mathbf{z(t) = A\,e^{iwt}}$; $z2(t) = \mathbf{Be^{(iwt+\Phi)}}$; phi= phase angle
- $Re(z1(t)) = A\cos(wt\ )$; $Re(z2(t)) = B\cos(wt + \Phi)$



Aeiwt rotates with angular frequency w in the complex plane.

Oscillatory complex variable

- When the phase angle is positive, the second "leads" the first, when negative the second "lags".

- z1 and z2, are rotate with time at the same angular frequency, but there is a difference in magnitude and phase angle . This can be captured on a static diagram by choosing a particular value of time at which to "freeze" the quantities. Such frozen quantities are call phasors, and the diagram called a **phasor diagram.**

Phasor Diagram

- The exact choice of time doesn't matter, but it is often convenient to choose a time which makes one of the frozen phasors lie along the real axis — this defines the reference phase.
- Eg: if we choose t = 0, then z1 becomes the reference phase,

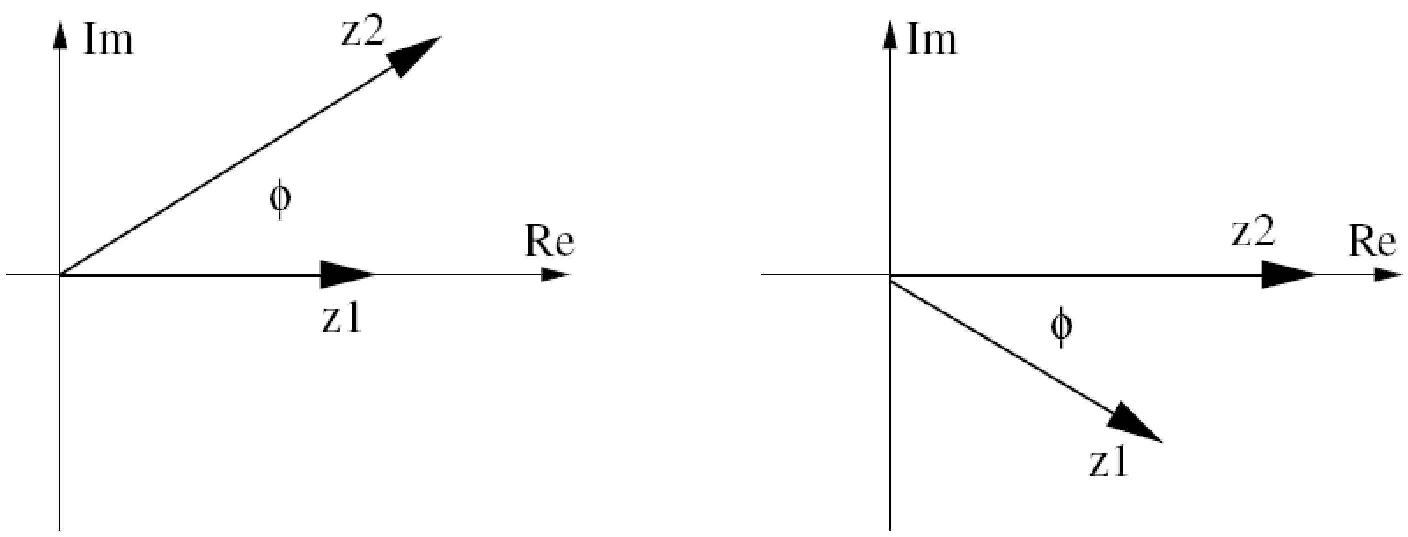


Figure 3.3: Phasor Diagrams using different reference phases

An Application of Oscillatory Complex Numbers:

AC Circuits

Phasor Diagram

The impedance of an ac circuit is the total effective resistance to the flow of current by a combination of the elements of the circuit.

$$Z=R+j(XL-XC)$$

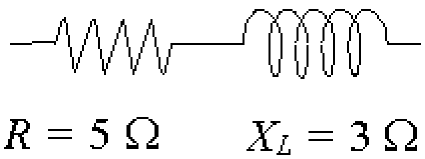
$$|Z|=\sqrt{}R2+(XL-XC)2$$

- **Phase angle**

- tan $\theta=(R/XL-XC)$
- Θ = tan_1(R/XL-XC)

Phasor Diagram

Ex:1 : A circuit has a resistance of 5 Ω in series with a reactance across an inductor of 3 Ω. Represent the impedance by a complex number, in polar form.



- Ans: XL=3 Ω and XC=0 so XL−XC=3 Ω.
- Z=5+3j Ω ; |z| =5.83,
- Φ (the phase difference) : 30.96
- Z=5.83∠30.96∘ Ω.

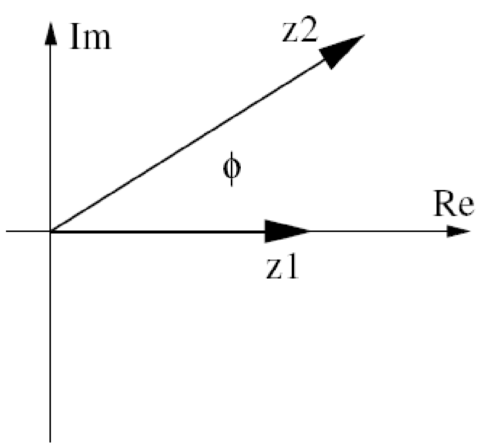
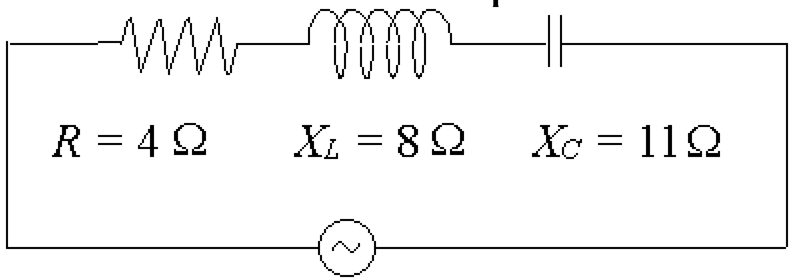


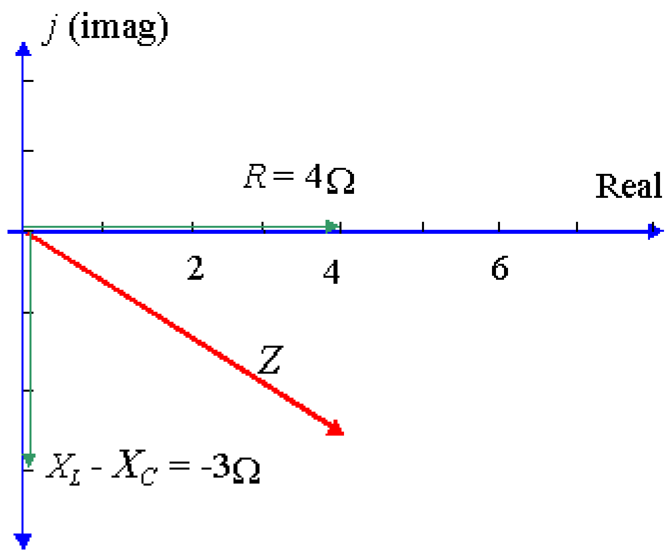
Figure 3.3: Phasor Diagrams

- **Example 2(a)**

A particular ac circuit has a resistor of 4 Ω, a reactance across an inductor of 8Ω and a reactance across a capacitor of 11Ω. Express the impedance of the circuit as a complex number in polar form.



- $X_L - X_C = 8 - 11 = -3\ \Omega$
- So $Z = 4 - 3j\ \Omega$; in rectangular form
- $r = 5$ and $\theta = -36.87°$.
- Polar form: $Z = 5\angle -36.87°\ \Omega$



Thank you,,,,,